SELF-SIMILARITY PROBLEM IN HIGH-TEMPERATURE FLOWS AND MINIMAL GRADIENT PRINCIPLE

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The problem

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial v_z}{\partial r}=\rho\left(v_r\frac{\partial v_z}{\partial r}+v_z\frac{\partial v_z}{\partial z}\right),\ \frac{1}{r}\frac{\partial}{\partial r}r\rho v_r+\frac{\partial}{\partial z}\rho v_z=0,$$
$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial T}{\partial r}=\Pr\rho\left(v_r\frac{\partial T}{\partial r}+v_z\frac{\partial T}{\partial z}\right),\ \rho T=1,\ v_r=\frac{\partial v_z}{\partial r}=\frac{\partial T}{\partial r}=0 \text{ for } r=0; \tag{1}$$
$$v_z=0,\ T=T_\infty \text{ for } r\to\infty \tag{2}$$

with the conditions

$$\int_0^\infty \rho v_z^2 r\,dr=1,\ \int_0^\infty \rho v_z(T-T_\infty)r\,dr=1. \tag{3}$$

occurs in the investigation of a high-temperature jet flow out of a cylindrical orifice in a boundary-layer approximation. Here r, zR are cylindrical coordinates (r and z are the internal coordinates in an asymptotic expansion in the small parameter R^{-1}); $R=\sqrt{\rho_m I_{1m}/2\pi}/\mu_m$ is the analog of the Reynolds number; v_z and $v_r R$ are the axial and radial velocity components; T, temperature; ρ, density; and $\Pr = c_{pm}\mu_m/\lambda_m$, the Prandtl number. The temperature scale T_m, density ρ_m, specific heat at constant pressure c_{pm}, heat conductivity λ_m, dynamic viscosity μ_m, total momentum flux I_{1m}, total enthalpy flux I_{2m}:

$$I_{1m}=2\pi\rho_m V_m^2 L_m^2\int_0^\infty \rho v_z^2 r\,dr,$$
$$I_{2m}=2\pi c_{pm}\rho_m T_m V_m L_m^2\int_0^\infty \rho v_z(T-T_\infty)r\,dr$$

are considered given. Selected, respectively, as velocity and length scales are $V_m = c_{pm}\cdot T_m I_{1m}/I_{2m}$, $L_m = I_{2m}/(c_{pm}T_m\sqrt{2\pi\rho_m I_{1m}})$.

As $T_\infty \to 0$ the boundary conditions (2) and the integral conditions (3) can be rewritten in the form [1]

$$\int_0^{r_0(z)} \rho v_z^2 r\,dr=1,\ \int_0^{r_0(z)} v_z r\,dr=1,\ v_z=T=0 \text{ for } r\to r_0(z), \tag{4}$$

where the integrals are written down under the assumption of their existence, and $r_0(z)$ is the interfacial surface separating the high-temperature compressed gas flow from the cold incompressible gas flow with constant temperature. Let us note that the interfacial surface cannot exist in a certain range of parameters and then $r_0 \to \infty$ should be presumed in (4). As $T_\infty \to 0$, let us construct the self-similar solution for the problem (1), (4)

$$v_z(r,z)=\frac{3-\Pr}{4}\frac{1}{z}\left(1-\frac{x^2}{x_0^2}\right)^{2/(\Pr-1)},\ T=\frac{\Pr+1}{4}\frac{1}{z}\left(1-\frac{x^2}{x_0^2}\right)^{2\Pr/(\Pr-1)},$$
$$v=\frac{3-\Pr}{8}\frac{r}{z^2}\left[\left(1-\frac{x^2}{x_0^2}\right)^{2/(\Pr-1)}-\left(1-\frac{x^2}{x_0^2}\right)^{(1+\Pr)/(\Pr-1)}\right],$$

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$$x = r/\sqrt{z}, \quad x_0^2 = 8(\mathrm{Pr}+1)/[(3-\mathrm{Pr})(\mathrm{Pr}-1)], \tag{5}$$

which is suitable for the semi-infinite interval r ($0 \le r < \infty$) for Pr < 1 and, as is shown in [1], is suitable in the finite interval $0 \le r < r_0(z)$ for $1 \le \mathrm{Pr} < 3$. The solution (5) becomes unsuitable for Pr > 3 since the self-similarity constant for the velocity is obtained under the assumption of existence of integrals (4) while for Pr ≥ 3 the first integral in (4) does not exist.

For Pr > 3 we seek the solution of the problem (1), (4) in the form

$$v_z(r, z) = z^{\alpha_w} w(x), \quad T = z^{-1}\theta(x), \quad v_r = z^{-\alpha_r - 1} v(x),$$
$$r = z^{\alpha_r} x, \quad \alpha_w = -2\alpha_r. \tag{6}$$

Here w, θ, v, and x are self-similar vairables; α_w and α_r are self-similarity constants, and the existence of the second integral in (4) is assumed in establishing relations between the self-similarity constants. Substituting (6) into (1) and (4), we obtain

$$\frac{1}{x}(xw')' = \frac{1}{\theta}[vw' + w(\alpha_w w - \alpha_r x w')],$$
$$\frac{1}{x}\left(\frac{xv}{\theta}\right)' + (\alpha_w + 1)\frac{w}{\theta} - \alpha_r x\left(\frac{w}{\theta}\right)' = 0, \tag{7}$$
$$\frac{1}{x}(x\theta')' = \mathrm{Pr}\,\theta^{-1}[v\theta' + w(-\theta - \alpha_r x\theta')],$$
$$w' = \theta' = v = 0 \text{ for } x = 0, \; w = \theta = 0 \text{ for } x = x_0$$

(x_0 is the separation point related to the interfacial surface by the formula $r_0 = z^{\alpha_r} x_0$). Introducing the new variable

$$s = \mathrm{Pr}(v - \alpha_r x w), \tag{8}$$

we convert the problem (7) to

$$\frac{1}{x}(xw')' = \frac{sw'}{\mathrm{Pr}\,\theta} + \alpha_w \frac{w^2}{\theta}, \quad \frac{1}{x}(xs)' = \frac{s\theta'}{\theta} - \mathrm{Pr}\,w,$$
$$\frac{1}{x}(x\theta')' = \frac{s\theta'}{\theta} - \mathrm{Pr}\,w, \; w' = \theta' = s = 0 \text{ for } x = 0, \; w = \theta = 0 \text{ for } x = x_0. \tag{9}$$

Since the equations and initial conditions for (9) agree in s and θ', then

$$s = \theta'. \tag{10}$$

Therefore, the order can be reduced in problem (9), and by using (10) it takes the form

$$\frac{1}{x}(xw')' = \frac{\theta' w'}{\mathrm{Pr}\,\theta} + \alpha_w \frac{w^2}{\theta}, \quad \frac{1}{x}(x\theta')' = \frac{\theta'^2}{\theta} - \mathrm{Pr}\,w,$$
$$w' = \theta' = 0 \text{ for } x = 0, \; w = \theta = 0 \text{ for } x = x_0. \tag{11}$$

Let us note that the solutions of problem (11) are invariant to the transformation

$$w \to C_1 w, \quad \theta \to C_2\theta, \quad x \to C_1^{-1/2} C_2^{1/2} x \tag{12}$$

(C_1 and C_2 are arbitrary constants). This permits giving the initial conditions, say

$$w = \theta = 1 \text{ for } x = 0, \tag{13}$$

as nontriviality conditions. Solving problem (11), (13) and using the invariant properties (12), the solutions can be normalized in conformity with the integral conditions (4) or in some other manner.

It could be expected that the solutions of problem (11), (13) exist for not every value of the self-similarity constants for the velocity α_w (i.e., α_w is the analog of an eigenvalue) but it turns out in a numerical computation that α_w takes on a set of values from the interval $\alpha_{we} < \alpha_w < \infty$. In order to formulate the condition from which to start to be able to select a unique value of α_w, we examine solutions of the problem (11), (13) in the neighborhood of the separation point

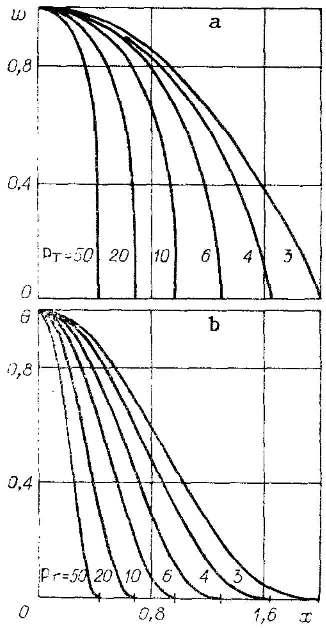

Fig. 1

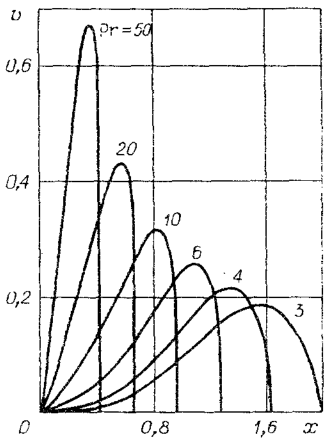

Fig. 2

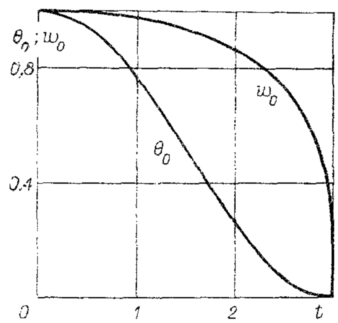

Fig. 3

$$w = A(x_0 - x)^a, \; \theta = B(x_0 - x)^b. \tag{14}$$

Substituting (14) into (11) we have

$$B[a(a-1) - ab\,\mathrm{Pr}^{-1}] - A\alpha_w = 0, \;\; Bb - A\,\mathrm{Pr} = 0, \;\; b = a + 2. \tag{15}$$

Equations (15) are linear and homogeneous in the unknowns A and B. Equating the determinant of system (15) to zero, we obtain

$$a = \frac{\mathrm{Pr} + 2 + \alpha_w}{2(\mathrm{Pr} - 1)} \pm \frac{1}{2(\mathrm{Pr} - 1)} \sqrt{\mathrm{Pr}^2 + (4 + 10\alpha_w)\,\mathrm{Pr} + (\alpha_w - 2)^2},$$
$$B = \frac{\mathrm{Pr}}{a + 2} A. \tag{16}$$

It follows from numerical computations of the problem (11), (13) that its solutions in the neighborhood of the separation point behave in conformity with (14), (16), where the minus sign must be taken for the square root in the formula for a in (16).

Therefore, a continuum of solutions (6) satisfying system (1) and boundary conditions (1) and (4) exists. It is evident from physical considerations that the viscosity hinders flows with high velocity gradients (the equivalent can be shown for the influence of heat conductivity on the temperature gradient). Consequently, we formulate the following principle as the selection criterion for the solutions: out of the mathematically possible high-temperature flows only that for which the velocity gradient (or equivalently in this case, the temperature gradient) is minimal at the interfacial surface is realized physically. In connection with the fact that the velocity gradient on the interfacial surface is unlimitedly high, it would terminologically be stricter to require the minimum of the velocity-gradient singularity and the minimum of the singularity of any derivative that tends to infinity for the temperature. However, the physical content is reflected best in the terminology "minimal gradient" (it should not be examined on the interfacial surface itself but in an arbitrarily nearby neighborhood), and even more so since situations apparently exist when neither the velocity (or temperature) gradient nor its derivative in the neighborhood of the interfacial surface have singularities. The minimal gradient principle is used for the problem under consideration only if integrals of the form (4) do not exist.

The results from (16) and numerical computations that for Pr > 3 the velocity gradient at the interfacial surface is minimal (a is maximal) if

$$\alpha_w = 2-5\mathrm{Pr} + \sqrt{24\mathrm{Pr}(\mathrm{Pr}-1)}, \tag{17}$$

the value of a is here determined from the formula

$$a = \sqrt{6\mathrm{Pr}/(\mathrm{Pr}-1)} - 2, \tag{18}$$

i.e., the minimal gradient principle is realized as the multiplicity condition for the root a in (16). Within the framework of the present investigation the minimal gradient principle denotes the selection of the limit value of α_w below which the solution of problem (11), (13) will already not exist from the mathematical viewpoint. There results from (18) that $a < 1$ for Pr > 3, i.e., the velocity gradient at the interfacial surface is unlimitedly large. In connection with the multiplicity of the root a in (16) for system (11), both solutions of the form (14) and the solutions $w = A(x_0 - x)^a \ln(x_0 - x)$, $\theta = B(x_0 - x)^b \ln(x_0 - x)$ are applicable, and it is difficult to establish numerically which will be realized near the separation point.

Numerically constructed solutions of problem (11), (13) and (17) are displayed in Fig. 1 for an arbitrary velocity and temperature for different Pr, while the solution for the transverse velocity v, determined by using (10) and (8), is displayed in Fig. 2. As Pr increases, the longitudinal velocity profile becomes more inflated, the high-temperature boundary-layer thickness x_0 diminishes, and the transverse velocity component increases. As $\mathrm{Pr} \to \infty$ by formulating the limit process in the form

$$t = x\sqrt{\mathrm{Pr}}, \tag{19}$$

which is fixed as $\mathrm{Pr} \to \infty$, we construct the asymptotic expansion

$$\begin{gathered} w(x, \mathrm{Pr}) = w_0(t) + \ldots,\ \theta(x, \mathrm{Pr}) = \theta_0(t) + \ldots, \\ \alpha_w(\mathrm{Pr}) = \alpha_0 \mathrm{Pr} + \ldots,\ x_0(\mathrm{Pr}) = t_0 \mathrm{Pr}^{-1/2} + \ldots \end{gathered} \tag{20}$$

[$\alpha_0 = -5 + \sqrt{24}$ (17)]. Substituting (20) into (11), (13), we obtain in the limit (19) in the zeroth approximation in Pr^{-1}

$$\begin{gathered} \frac{1}{t}\frac{d}{dt}t\frac{dw_0}{dt} = \alpha_0\frac{w_0^2}{\theta_0},\quad \frac{1}{t}\frac{d}{dt}t\frac{d\theta_0}{dt} = \frac{1}{\theta_0}\left(\frac{d\theta_0}{dt}\right)^2 - w_0, \\ \frac{dw_0}{dt} = \frac{d\theta_0}{dt} = 0,\quad w_0 = \theta_0 = 1 \text{ for } t = 0,\quad w_0 = \theta_0 = 0 \text{ for } t = t_0. \end{gathered} \tag{21}$$

The solutions of problem (21) are shown in Fig. 3. Therefore, the fullness of the profile w is bounded by the function w_0 as $\mathrm{Pr} \to \infty$.

A numerical experiment was performed to confirm the minimal gradient principle: the partial derivative problem (1), (2) was supplemented by the necessary initial conditions for $z = z_0$ and was solved numerically (in finite differences) for sufficiently small values of T_∞. These computations showed that beyond the dependence on the initial conditions the solutions became self-similar quite rapidly as z grew, where the self-similarity constants satisfied the minimal gradient principle with sufficient accuracy.

The Runge–Kutta method was used for numerical computations of problems with ordinary differential equations and the factorization method with iterations for the partial differential problems, and the idea of perturbation methods [2] for the construction of the asymptotic expansions.

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STABILITY OF THERMOCAPILLARY MOTION IN A CYLINDRICAL LAYER

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The stability of thermocapillary motion in a planar layer and a liquid cylinder was studied in [1, 2]. In the present study we will consider the stability of thermocapillary convection in a cylindrical layer with an undeformed free surface. The effect of the ratio of cylinder radii on motion stability is considered. It is shown that for axisymmetric disturbances at certain values of the problem parameters, increase in relative thickness of the inner cylinder leads to reduction in stability.

1. We will consider a cylindrical layer of viscous thermally conductive liquid bounded by solid inner and free outer surfaces in the absence of gravity. We introduce a cylindrical coordinate system with the z axis directed along the cylinder directrix. The equation of the solid boundary is $r = r_0$. We assume that the free surface is cylindrical ($r = r_1$) and undeformed. The temperature dependence of the surface tension coefficient is given by $\sigma = \sigma_0 - \kappa(\theta - \theta_0)$.

Let the free surface be heated by a law $\theta_B = -Az$ (A is a specified constant value). Then the steady-state axisymmetric thermocapillary motion which develops due to change in surface tension will be described by the equations

$$\begin{gathered} u = v = 0,\ w = B_1(\xi^2 - d^2) + B_2 \ln(\xi/d),\ p_\eta = 4B_1, \\ \theta = -\eta - \mathrm{MaPr}\,[B_1(\xi^4 - 1)/4 - (d^2B_1 + B_2 + \ln dB_2)(\xi^2 - 1) + \\ + B_2(\xi^2 + d^2)\ln\xi + B_1 d^4 \ln\xi]/4, \end{gathered} \tag{1.1}$$

where the constants $B_1 = (1 - d^2 + 2\ln d)[(1 - d^2)(3 - d^2) + 4\ln d]^{-1}$, $B_2 = (1 - d^2)^2[(1 - d^2)(3 - d^2) + 4\ln d]^{-1}$ are found from the conditions of adhesion and closed flow

$$\int_d^1 \xi w(\xi)\, d\xi = 0. \tag{1.2}$$

Here and below, $\xi = r/r_1$; $\eta = z/r_1$; $d = r_0/r_1 < 1$; $\mathrm{Ma} = r_1{}^2\kappa A/\rho\nu^2$ is the Marangoni number; $\mathrm{Pr} = \nu/\chi$, the Prandtl number; $\mathrm{Bi} = \beta r_1/\lambda$, the Biot number; ν and χ, kinematic viscosity and thermal diffusivity coefficients; λ and β, thermal conductivity and interphase exchange coefficients; ρ, density. For units of length, time, velocity, temperature, and pressure we take r_1, $r_1{}^2/\nu\mathrm{Ma}$, $\nu\mathrm{Ma}/r_1$, Ar_1, and $\rho\nu^2\mathrm{Ma}^2/r_1{}^2$, respectively.

As $d \to 0$ the motion of Eq. (1.1) transforms to thermocapillary flow of a completely liquid cylinder: $u = v = 0$, $w = (\xi^2 - 0.5)/2$, $p_\eta = 2$, $\theta = -\eta - \mathrm{MaPr}(1 - \xi^2)^2/32$, the stability of which was studied in [2]. In [3] a stability study was performed for axisymmetric disturbances of a motion with logarithmic velocity profile which did not satisfy closure condition (1.2).

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